

Tables of Complete Elliptic Integrals¹

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The complete elliptic integrals

$$K = \int_0^{\pi/2} (1 - k^2 \sin^2 \phi)^{-\frac{1}{2}} d\phi,$$

$$E = \int_0^{\pi/2} (1 - k^2 \sin^2 \phi)^{\frac{1}{2}} d\phi$$

and the associated function

$$M = \frac{\pi}{2K}$$

have been tabulated extensively^{3,4} with k^2 and k and

¹ This paper was prepared under a National Bureau of Standards contract with American University.

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³ A. Fletcher, J. C. P. Miller, and L. Rosenhead, An index of mathematical tables (Scientific Computing Service, London, 1946).

⁴ A. Fletcher, Guide to tables of elliptic functions, MTAC 3, 229-281 (1948).

$\arcsin k$ as arguments. This replication of tables with a variety of closely related arguments is partly due to the difficulty of interpolating in the neighborhood of $k=1$. Legendre considered that interpolation was unsatisfactory in the range $1 \leq 1/k \leq \sqrt{2}$. Recently the author needed tables of K and E with $1/k$ as argument. This is quite a common need. For example, it arises in evaluating integrals of the form

$$\int_0^1 \frac{dx}{\sqrt{(1-x^2)(p^2-x^2)}}, \quad pk=1.$$

The tables, given below, were computed by Gauss's arithmetic-geometric mean. The last significant figure is doubtful, but is retained in accordance with Fisher and Wishart.⁵

⁵ R. A. Fisher and J. Wishart, On the distribution of the error of an interpolated value, and on the construction of tables, Proc. Cambridge Phil. Soc. 23, 912-921 (1927).

$1/k$	K	E	M
1. 00	∞	1.	0.
1. 01	3. 36145 8120	1. 02824 2731	0. 46729 61170
1. 02	3. 03047 9311	1. 04930 1697	0. 51833 26351
1. 03	2. 84195 7373	1. 06755 5825	0. 55271 63573
1. 04	2. 71141 0062	1. 08395 7261	0. 57932 82058
1. 05	2. 61243 9112	1. 09896 6268	0. 60127 57656
1. 06	2. 53331 9380	1. 11285 8448	0. 62005 45970
1. 07	2. 46781 4067	1. 12581 8774	0. 63651 32398
1. 08	2. 41221 1023	1. 13798 0652	0. 65118 52868
1. 09	2. 36412 2792	1. 14944 5208	0. 66443 09393
1. 10	2. 32192 4037	1. 16029 1946	0. 67650 63378
1. 11	2. 28445 8951	1. 17058 5138	0. 68760 10297
1. 12	2. 25087 6725	1. 18037 7893	0. 69785 97760
1. 13	2. 22053 3243	1. 18971 4883	0. 70739 59970
1. 14	2. 19292 9400	1. 19863 4233	0. 71630 04550
1. 15	2. 16767 0799	1. 20716 8882	0. 72464 70855
1. 16	2. 14444 0528	1. 21534 7583	0. 73249 70342
1. 17	2. 12298 0225	1. 22319 5658	0. 73990 15350
1. 18	2. 10307 6584	1. 23073 5581	0. 74690 40066
1. 19	2. 08455 1532	1. 23798 7432	0. 75354 16145
1. 20	2. 06725 4932	1. 24496 9258	0. 75984 64529
1. 21	2. 05105 9081	1. 25169 7363	0. 76584 64553
1. 22	2. 03585 4498	1. 25818 6552	0. 77156 61060
1. 23	2. 02154 6642	1. 26445 0315	0. 77702 70023
1. 24	2. 00805 3348	1. 27050 1000	0. 78224 83048
1. 25	1. 99530 2778	1. 27634 9943	0. 78724 71008
1. 26	1. 98323 1784	1. 28200 7587	0. 79203 87014
1. 27	1. 97178 4578	1. 28748 3577	0. 79663 68863
1. 28	1. 96091 1645	1. 29278 6848	0. 80105 41070
1. 29	1. 95056 8854	1. 29792 5693	0. 80530 16554
1. 30	1. 94071 6711	1. 30290 7829	0. 80938 98084
1. 32	1. 92234 6001	1. 31243 0289	0. 81712 46622
1. 34	1. 90555 5240	1. 32140 6374	0. 82432 47395
1. 36	1. 89014 3668	1. 32988 1897	0. 83104 59960
1. 38	1. 87594 4456	1. 33789 7365	0. 83733 62664

$1/k$	K	E	M
1. 40	1. 86281 7607	1. 34548 8787	0. 84323 67835
1. 42	1. 85064 4602	1. 35268 8327	0. 84878 33512
1. 44	1. 83932 4316	1. 35952 4838	0. 85400 72640
1. 46	1. 82876 9850	1. 36602 4314	0. 85893 60368
1. 48	1. 81890 6041	1. 37221 0246	0. 86359 39910
1. 50	1. 80966 7496	1. 37810 3938	0. 86800 27302
1. 52	1. 80099 7013	1. 38372 4769	0. 87218 15280
1. 54	1. 79284 4308	1. 38909 0414	0. 87614 76498
1. 56	1. 78516 4971	1. 39421 7036	0. 87991 66198
1. 58	1. 77791 9616	1. 39911 9448	0. 88350 24444
1. 60	1. 77107 3176	1. 40381 1262	0. 88691 78010
1. 62	1. 76459 4312	1. 40830 5004	0. 89017 41982
1. 64	1. 75845 4921	1. 41261 2228	0. 89328 21127
1. 66	1. 75262 9723	1. 41674 3613	0. 89625 11056
1. 68	1. 74709 5907	1. 42070 9039	0. 89908 99244
1. 70	1. 74183 2833	1. 42451 7669	0. 90180 65896
1. 72	1. 73682 1775	1. 42817 8011	0. 90440 84715
1. 74	1. 73204 5702	1. 43169 7974	0. 90690 23555
1. 76	1. 72748 9086	1. 43508 4926	0. 90929 45015
1. 78	1. 72313 7737	1. 43834 5734	0. 91159 06946
1. 80	1. 71897 8667	1. 44148 6811	0. 91379 62892
1. 82	1. 71499 9953	1. 44451 4152	0. 91591 62509
1. 84	1. 71119 0637	1. 44743 3363	0. 91795 51902
1. 86	1. 70754 0629	1. 45024 9701	0. 91991 73949
1. 88	1. 70404 0617	1. 45296 8092	0. 92180 68582
1. 90	1. 70068 1997	1. 45559 3163	0. 92362 73036
1. 92	1. 69745 6806	1. 45812 9262	0. 92538 22079
1. 94	1. 69435 7662	1. 46058 0475	0. 92707 48213
1. 96	1. 69137 7713	1. 46295 0655	0. 92870 81858
1. 98	1. 68851 0587	1. 46524 3426	0. 93028 51514
2. 00	1. 68575 0355	1. 46746 2209	0. 93180 83916

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